Dr. Stefano Filipazzi
Dr. Alapan Mukhopadhyay
Léo Navarro Chafloque

EPFL, fall semester 2024
AG II - Schemes and sheaves

Solutions – week 5

We strongly advise the reader to go check the *blow-ups* document on moodle.

Exercise 3. *Blow-ups*

(1) We show that b induces an isomorphism

$$b\colon b^{-1}(U) \to U.$$

To see this, let $f \in I$ so that $D(f) \subset U$. Note that $I_f = R_f$. Therefore by the compatibility of Proj and pullbacks we have as in the example above

$$b^{-1}(U) = \mathrm{Proj}(\bigoplus_{n\geq 0} R_f) = \mathrm{Proj}(R_f[t]) = \mathrm{Spec}(R_f).$$

As U is covered by such $D(f)$'s the above map is locally an isomorphism and therefore an isomorphism.

(2) Note that I^n/I^2 is a free A-module of rank $n+1$ generated by $x_0, \dots, x_n$. More generally I^n/I^{n+1} is a free A-a module generated by degree n-monomials. This leads to a graded isomorphism

$$\bigoplus_n I^n/I^{n+1} \cong A[x_0, \dots, x_n].$$

(3) We proof something more general.

Definition 0.1 (Regular sequence)**.** Let R be a ring. A finite sequence of elements $f_1, \dots, f_n$ is said to be a *regular sequence* if f_i is a non-zero divisor in $R/(f_1, \dots, f_{i-1})$ and $R/(f_1, \dots, f_n)$ is non-zero.

We show the following.

Proposition 0.1. *Let R be a ring and $I = (f_1, \dots, f_n)$ where $f_1, \dots, f_n$ form a regular sequence. Then the kernel of the surjection sending Y_i to $f_i^{(1)}$*

$$R[Y_1, \dots, Y_n] \to \bigoplus_{n\geq 0} I^n$$

is given by the ideal $J = (f_iY_j - f_jY_j)$.

Proof. We show this using two steps which both really heavily on the regular sequence hypothesis.

First, we show that the kernel of

$$R^n \to I$$

sending $e_i \mapsto f_i$ is generated by the vectors $e_if_j - e_jf_i$. We proceed by induction on n the length of the regular sequence. For $n = 0, 1$ the claim is obvious. To proceed inductively we define the chain complex K_n

$$R^{\binom{n}{2}} \to R^n \to R$$

with R placed in degree zero and differentials being respectively given by $e_{i,j} \mapsto f_j e_i - f_j e_i$ and $e_i \mapsto f_i$. Note that $H_0(K_n)$ of this complex is $R/(f_1, \dots, f_n)$. and that the claim amounts to this complex being exact in the middle, meaning that $H_1(K_n) = 0$. Note that we also have the following exact sequence of complexes

$$\begin{tikzcd}
R^{\binom{n}{2}} \arrow[r] \arrow[d] & R^n \arrow[r] \arrow[d] & R \arrow[d] \\
R^{\binom{n+1}{2}} \arrow[r] \arrow[d] & R^{n+1} \arrow[r] \arrow[r, shift right, red] \arrow[d] & R \arrow[d] \arrow[u, shift right, red] \\
R^n \arrow[r] & R \arrow[r] \arrow[u, shift right, red] & 0
\end{tikzcd}$$

Where the left vertical arrows are given by $e_{i,j} \mapsto e_{i,j}$ and $e_{i,j} \mapsto \delta_{n+1,j} e_i$. The middle vertical arrows are $e_i \mapsto e_i$ and $e_i \mapsto \delta_{i,n+1}$. The first right vertical arrow is the identity.

By induction $H_1(K_n) = 0$, and we want to show that $H_1(K_{n+1}) = 0$. The long exact sequence in homology gives

$$0 = H_1(K_n) \to H_1(K_{n+1}) \to R/(f_1, \dots, f_n) \xrightarrow{\delta} R/(f_1, \dots, f_n),$$

where δ is the connecting morphism. The connecting morphism is computed by following the red arrows on the diagram above. It is therefore given by $\delta = \cdot f_{n+1}$ the multiplication by f_{n+1}. As $f_1, \dots, f_{n+1}$ is a regular sequence δ is injective and therefore $H_1(K_{n+1}) = 0$.

We have now understood the degree 1 elements of the kernel of the surjection sending Y_i to $f_i^{(1)}$

$$R[Y_1, \dots, Y_n] \to \bigoplus_{n \geq 0} I^n.$$

It now suffices to show that this kernel is generated by degree 1 elements.

Just for the rest of this proof, we call a polynomial $F \in R[Y_1, \dots, Y_n]$ to be of *weight* i if i is the minimal integer such that $F \in (Y_1, \dots, Y_i)$ but $f(Y_1, \dots, Y_n) \notin (Y_1, \dots, Y_{i-1})$. A weight 0 polynomial is defined to be 0.

This proposition will be shown as a special case (but the general case will be needed in the proof by induction) of the following.

Claim. *Let $F \in R[Y_1, \dots, Y_n]$ be an homogeneous polynomial of degree m with*

$$F(f_1, \dots, f_n) \in (f_1, \dots, f_k).$$

Then there exists an homogeneous polynomial G of degree m and weight at most k such that $F - G \in (f_i Y_j - f_j Y_j)$. In particular if $F(f_1, \dots, f_n) = 0$, then $F \in J = (f_i Y_j - f_j Y_j)$, showing the proposition.

We prove the claim by induction on the degree m of the polynomial. Let F be a polynomial of degree 1. Then

$$F(f_1, \dots, f_n) = \sum_{i=1}^{k} a_i f_i.$$

Therefore the weight k and degree 1 polynomial $G = \sum_{i=1}^{k} a_i Y_i$ satisfies the claim: indeed $F - G$ is an homogeneous polynomial of degree 1 with $(F-G)(f_1, \dots, f_n) = 0$. Therefore using the first part of the proof above, we see that $F - G \in J$.

Now if F is a polynomial of degree m, we show the claim by induction on the weight l of F. If $l \leq k$, set $F = G$. Otherwise write $F = Y_l F_1 + F_2$ with F_1 homogeneous of degree $m-1$ and F_2 of weight at most $l-1$. Recall that by hypothesis

$$f_l F_1(f_1, \dots, f_n) + F_2(f_1, \dots, f_n) \in (f_1, \dots, f_k) \subset (f_1, \dots, f_{l-1})$$

and $F_2(f_1, \dots, f_n) \in (f_1, \dots, f_{l-1})$ because F_2 is of weight at most $l-1$ by construction. Modding out by $f_1, \dots, f_{l-1}$ and using that $f_1 \dots, f_l$ is a regular sequence we get that $F_1(f_1, \dots, f_n) \in (f_1, \dots, f_{l-1})$. We apply induction on the degree to get a polynomial G_1 of weight at most $l-1$ such that $F_1 - G_1 \in J$. Now set $G' = Y_l G_1 + F_2$. This is a polynomial of weight at most $l-1$ because G_1 and F_2 are. Note also that $F - G' = Y_l(F_1 - G_1) \in J$. Note also that $G'(f_1, \dots, f_n) = F(f_1, \dots, f_n) \in (f_1, \dots, f_k)$. By induction on the weight there is a polynomial G of weight at most k such that $G' - G \in J$. But now $F - G = (F - G') + (G' - G) \in J$, concluding the proof. □

Exercise 4. *Strict transforms.*

(1) By definition the strict transform St_J is

$$\mathrm{Proj}(\bigoplus_{n \geq 0} (I+J)^n / J)$$

As the kernel of $I^n \to (I+J)^n/J$ is $I^n \cap J$ we see that we can realize the strict transform as the closed subscheme of Bl_I given by $V_+(\bigoplus_{n \geq 0} I^n \cap J)$.

(2) This is similar to exercise 3, point 1.

(3) Let k be a field and consider $R = k[x_0, x_1]$, $I = (x_0, x_1)$ and $J = (x_1^2 - (x_0^3 + x_0^2))$ which is a singular plane curve, which is the called the *node*. We compute the strict transform St_J. We claim that $\mathrm{St}_J \cong \mathbb{A}_k^1$ (which is regular) and that the blow-up map may be described as $\mathbb{A}_k^1 \to C \subset \mathbb{A}_k^2$

$$\lambda \mapsto (\lambda^2 - 1, \lambda^3 - \lambda).$$

We use the standard charts, meaning that we see $\mathrm{Bl}_I \subset \mathbb{A}^2 \times \mathbb{P}^1$. Recall that this inclusion is induced by the surjection

$$k[x_0, x_1, Y_0, Y_1] \to \bigoplus_n I^n$$

sending Y_i to x_i in degree 1. We claim that the preimage of the ideal

$$V_+\left(\bigoplus_n I^n \cap J\right)$$

by the above map is given by $(H, (x_1Y_0 - x_0Y_1))$ where

$$H = (x_1^2 - (x_0^3 + x_0^2), x_1Y_1 - (x_0^2Y_0 + x_0Y_0), Y_1^2 - (x_0Y_0^2 + Y_0^2))$$

Indeed, for degree zero, one and two, these elements are sent to the generator of J (we have $I^n \cap J = J$ for $n \leq 2$).

We now argue that these generators are enough. Note that being in I^n for a polynomial means that the monomials forming it are at least of degree n. Being in J means that the polynomial is of the form $f(x_0, x_1)(x_1^2 - (x_0^3 + x_0^2))$ for an $f(x_0, x_1) \in k[x_0, x_1]$. So we see that if such an element $f(x_0, x_1)(x_1^2 - (x_0^3 + x_0^2))$ is in $I^n \cap J$, then $f(x_0, x_1) \in I^{n-2}$ counting the degrees of the monomials because $(x_1^2 - (x_0^3 + x_0^2)) \in I^2 \setminus I^3$. Therefore for $n \geq 3$ using the degree 2 generator and elements of I in degree 1, we can attain every element of $I^n \cap J$.

Therefore the strict transform is

$$\mathrm{Proj}(A[x_0, x_1, Y_0, Y_1]/(H, x_0Y_1 - x_1Y_0)).$$

Denote by B the grading ring we are taking Proj of. Note that $V_+(H, x_0Y_1 - x_1Y_0, Y_0) = \emptyset$, so that $V_+(H, x_0Y_1 - x_1Y_0, Y_0) \subset D_+(Y_0)$ implying that

$$\mathrm{Proj}(A[x_0, x_1, Y_0, Y_1]/(H, x_0Y_1 - x_1Y_0)) = \mathrm{Spec}(B_{(Y_0)}).$$

But, if we write $\frac{Y_1}{Y_0}$ by y we get

$$\begin{aligned} B_{(Y_0)} &= A[x_0, x_1, y]/(x_1^2 - (x_0^3 + x_0^2), x_1y - (x_0^2 + x_0), y^2 - (x_0 + 1), (x_0y - x_1)) \\ &\cong A[x_0, y]/(y^2 - (x_0 + 1)) \cong A[y]. \end{aligned}$$

Indeed using $x_0y = x_1$ the equation $x_1^2 - (x_0^3 + x_0^2)$ turn into $x_0^2(y^2 - (x_0 + 1))$ and $x_1y - (x_0^2 + x_0)$ turn into $x_0(y^2 - (x_0 + 1))$ which are both subsumed by the equation coming from degree 2.

Therefore we see that the strict transform is isomoprhic to $\mathbb{A}^1$. By using that under this isomorphisms $x_0 \mapsto y^2 - 1$ and $x_1 \mapsto x_0y = y^3 - y$ the claim about the form of the map follows.

Exercise 5.
We show the two lemmas mentioned in the exercise.

(a) We first treat the affine case $\mathrm{Spec}(A)$. If $g \in A$ is zero in A_f the by definition of localization we get that there is some n such that $f^ng = 0$. Now if X is quasi-compact, we cover X by finitely many open affines $(U_i)_{i=1}^n$ and find n_i such that $f^{n_i}g$ is zero when restricited to the respective open affines. Taking $N = \max\{n_i\}$ concludes by the sheaf property.

(1) We also first treat the affine case $X = \mathrm{Spec}(A)$. If $g \in A_f$ the there is some $n > 0$ such that f^ng is the image of an element $a \in A$. Now if X admits a cover as in the hypothesis, then we can find a_i in $\Gamma(U_i, \mathcal{O}_{U_i})$ such that a_i restricted to $U_{i,f}$ is $f^{n_i}g$. Arranging n and a_i by suitably

multiplying by power of f to get that such that a_i restricts to $f^n g$ on $U_{i,f}$. Now, $a_i - a_j \in \Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$ restricts to zero on $(U_i \cap U_j)_f$. Therefore by item (a), there is some n_{ij} such that $f^{n_{ij}}(a_i - a_j) = 0$ in $\Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j})$. Again, up to multiplying by some suitable power N we can replace a_i's so that they are compatible on intersections. Therefore by the sheaf property there is some $a \in \Gamma(X, \mathcal{O}_X)$ which satisfies what we want.